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Solving membrane vibration problem by the finite element interpolation post-processing on meshes with hanging nodes

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Abstract

This paper studies highly accurate algorithm for membrane vibration problem. Based on the theory of the interpolation post-processing, this paper establishes a scheme of bi-quadratic interpolation post-processing of bi-linear rectangular element on a locally refined mesh with hanging nodes. And the resulting solution is very satisfactory.

Keywords: finite element; membrane vibration problem; interpolation; Rayleigh quotient; hanging nodes

1. Introduction

During 1989 to 1991, Lin and Yang (see [1]) first pointed out that it is often possible to get the global stress super-convergence by higher degree interpolation for the finite element solution of lower order by using the nodes of lower order element as the interpolation nodes, which is called the finite element interpolation post-processing. Over the past 20 years, the finite element interpolation post-processing method has been developed greatly, and has been applied to many kinds of partial differential equations (see [2, 3, 4, and 5]).

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This paper establishes a scheme of finite element interpolation post-processing on locally refined mesh with hanging nodes. This scheme is very simple and feasible, and can be controlled automatically, thus, the computation costs, including the physical memory and the CPU time, will be reduced significantly. However, since there exists hanging nodes in locally refined mesh, the main work of this paper is to deal with hanging nodes.

We illustrate the efficiency of our scheme by the example of membrane vibration problem on the *L*-shaped region $\Omega=[0,1]\times[0,1]-[\frac{1}{2},1]\times[\frac{1}{2},1]$. We make a rectangular division for Ω and a locally refinement with hanging nodes near the singular point of Ω . We use bi-quadratic interpolation post-processing of bi-linear rectangular element to compute membrane vibration problem. And the resulting solution shows the efficiency of our scheme.

2. The finite element method of membrane vibrating problem

Consider the membrane vibrating problem:

$$\begin{aligned} &-\Delta u=\lambda u, \ in\,\Omega, \\ &u=0, \quad on\,\partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded region in R^2 , and its boundary $\partial\Omega$ is Lipschitz continuous.

The variational problem associated with (1) is given by: Find $\lambda\in R, u\in H_0^1(\Omega), u\neq 0$ satisfying

$$a(u,v)=\lambda b(u,v), \quad \forall\, v\in H_0^1(\Omega), \tag{2}$$

where $a(u,v)=\int_\Omega \nabla u\nabla v, b(u,v)=\int_\Omega uv.$

We take the L-shaped region $\Omega=[0,1]\times[0,1]-[\frac{1}{2},1]\times[\frac{1}{2},1]$ as an example to introduce the mesh generation K^h refined in local mesh with hanging nodes.

First let us decompose the *L*-shaped region $\Omega_0=\Omega$ (see Fig. 1. (a)) into a uniform grid consisting of many small rectangles whose diameters are $\frac{\sqrt{2}}{8}$, then each element in the sub-region $\Omega_1=[\frac{1}{4},\frac{3}{4}]\times[\frac{1}{4},\frac{3}{4}]-[\frac{1}{2},\frac{3}{4}]\times[\frac{1}{2},\frac{3}{4}]$ is subdivided into four identical rectangles whose diameters are $\frac{\sqrt{2}}{16}$, and then let us subdivide each element in the sub-region $\Omega_2=[\frac{3}{8},\frac{5}{8}]\times[\frac{3}{8},\frac{5}{8}]-[\frac{1}{2},\frac{5}{8}]\times[\frac{1}{2},\frac{5}{8}]$ into four identical rectangles whose diameters are $\frac{\sqrt{2}}{32}$. Repeating the procedure above, we can obtain a sequence of sub regions $\{\Omega_i\}$:

$$\Omega_i=\left[\frac{1}{2}(1-\frac{1}{2^i}),\frac{1}{2}(1-\frac{1}{2^i})\right]\times\left[\frac{1}{2}(1-\frac{1}{2^i}),\frac{1}{2}(1-\frac{1}{2^i})\right]-[\frac{1}{2},\frac{1}{2}(1+\frac{1}{2^i})]\times\left[\frac{1}{2},\frac{1}{2}(1+\frac{1}{2^i})\right]$$

and the mesh generation K^h refined in local mesh with hanging nodes, and this mesh is called regular locally refined mesh. We can obtain the locally refined mesh K^h (see Fig. 1. (b)) when Ω_0 is refined by 2 times. The black dots are hanging nodes and others as regular dots.

Let $S^h\subset H_0^1(\Omega)$ be a finite element space consisting of continuous piecewise bi-linear polynomials on K^h . Restricting the variational problem on S^h , we obtain discrete variational form (Finite Element Equation):

$$\begin{aligned} &\lambda_h\in R, u_h\in S^h, \qquad u_h\neq 0, \\ &a(u_h,v)=\lambda_h b(u_h,v), \forall\, v\in S^h. \end{aligned} \tag{3}$$

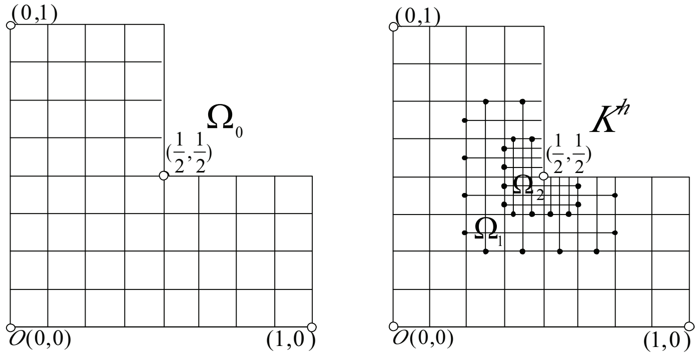

Fig. 1. (a) the initial mesh Ω_0 ; (b) the mesh K^h when Ω_0 is refined by 2 times.

3. The finite element interpolation post-processing

3.1. The bi-linear and the bi-quadratic interpolation of rectangular element with hanging nodes

As we know, for constrained finite element spaces on meshes with hanging nodes, if one uses the original basis functions to assemble stiff matrix and loaded matrix, then the resulting linear system (3) will like a “saddle system”. Hence it is not advisable to do this, so we must modify the basis functions.

Let u_i denote the value of u_h at node A_i , and φ_i the basis function of bi-linear interpolation at note A_i . If we solve (3), then

$$u_h = \sum_{i=1}^{n} u_i \varphi_i .$$

Restricting u_h on the edge $\overline{A_1 A_3}$ (see Fig. 2. (a)) we have

$$u_h = \sum_{i=1}^{3} u_i \varphi_i . \tag{4}$$

In order to keep u_h continuous on the edge $\overline{A_1 A_3}$, the value of u_h at hanging node A_2 should be satisfied the following condition:

$$u_2 = \frac{u_1 + u_2}{2} . \tag{5}$$

By substitution of (5) into (4), the basis function at hanging node A_2 will vanish automatically, then we get the new basis functions $\tilde{\varphi}_1$ and $\tilde{\varphi}_3$ at A_1, A_3 as follows:

$$\tilde{\varphi}_1 = \varphi_1 + \frac{1}{2}\varphi_2 ,\ \tilde{\varphi}_3 = \varphi_3 + \frac{1}{2}\varphi_2$$

then the continuity of u_h on edge $\overline{A_1 A_3}$ will be satisfied automatically.

For bi-quadratic interpolation of rectangular element, let ψ_i be the basis function of bi-quadratic interpolation on K^{2h} . To keep $I_{2h}^2 u_h$ continuous at hanging nodes A_2, A_4 (see Fig 2. (b)), u_h must satisfy the constraint condition

$$u_2 = \frac{3}{8}u_1 + \frac{3}{4}u_3 - \frac{1}{8}u_5 ,\ u_4 = -\frac{1}{8}u_1 + \frac{3}{4}u_3 + \frac{3}{8}u_5 . \tag{6}$$

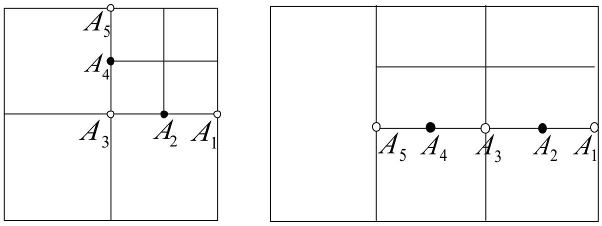

Fig. 2. (a) bi-linear interpolation with hanging nodes; (b) bi-quadratic interpolation with hanging nodes

Similar to the above, the basis functions at A_2 and A_4 will vanish when substituting (6) into $I_{2h}^2 u_h = \sum_{i=1}^{5} u_i \psi_i$. We can modify the basis functions ψ_1, ψ_3 and ψ_5 to ensure the continuity of $I_{2h}^2 u_h$ on the edge $\overline{A_1 A_5}$ automatically, that is

$$\tilde{\psi}_1 = \psi_1 + \frac{3}{8}\psi_2 - \frac{1}{8}\psi_4,\ \tilde{\psi}_3 = \psi_3 + \frac{3}{4}\psi_2 + \frac{3}{4}\psi_4,\ \tilde{\psi}_5 = \psi_5 - \frac{1}{8}\psi_2 + \frac{3}{8}\psi_4.$$

Based on the knowledge relating to bi-quadratic interpolation with hanging nodes above, we define interpolation operators I_h^1 and I_{2h}^2 by $I_h^1 : C^0(\Omega) \to S^h, I_{2h}^2 : C^0(\Omega) \to S_2^{2h}$ respectively, where S^h and S_2^{2h} are the finite element spaces of piecewise polynomial after modifying the basis functions at hanging nodes. I_h^1 and I_{2h}^2 satisfy the conditions of the following definition of interpolation post-processing operator. From now on, we write I_h^1 as I_h for simplicity.

3.2. The theory of the interpolation post-processing

Definition 1. Let K^h is a mesh over Ω. $S^h(\Omega)$ is a bi-m degrees finite element space on K^h, $I_h : C^0(\Omega) \to S^0(\Omega)$ is a piecewise bi-m degrees interpolation operator. $D \subset \Omega$ consists of some adjacent elements in K^h. We call I_h^j as j degree interpolation post-processing operator for m degree element $(j > m)$ if I_h^j meets the following conditions:

$$\begin{aligned} &\|I_h^j w - w\|_{s,p,D} \le Ch^{l+1-s}\|w\|_{l+1,p,D}, 2 \le p \le \infty, m < l \le j, \\ &I_h^j I_h v = I_h^j v, \\ &\|I_h^j v\|_{s,p,D} \le C\|v\|_{s,p,D}, \quad s = 0,1, \quad 2 \le p \le \infty, \quad \forall v \in S^h. \end{aligned}$$

where s is an positive integer, and C is independent of h, v and w.

Lemma 1. Let (λ, u) be an eigenpair of the problem (2), then for any $w \in H_0^1, \|w\|_b \ne 0$, the Rayleigh quotient of w satisfi

$$\frac{a(w,w)}{\|w\|_{0,2}} - \lambda = \frac{\|w-u\|_{1,2}^2}{\|w\|_{0,2}} - \lambda \frac{\|w-u\|_{1,2}^2}{\|w\|_{0,2}} \tag{7}$$

Proof. See [7].

Lemma 2. Suppose that $w \in H^{1+r_0-\varepsilon}(\Omega)$ where $\varepsilon > 0$ can be arbitrarily small, i.e. $w \sim \rho^{r_0}$. $\rho = \rho(x_1, x_2)$ is the distance between (x_1, x_2) and the singular point (0,0), $r_0 > 0$, and (0,0) is the only singular point. Let K^h be a regular mesh refined in local region by J times. S^h is a conforming bi-linear space on K^h, I_h is a bi-linear interpolation operator mapping onto S^h, then

$$a\left(w - I_h w, v\right) = (\vartheta(h^{\frac{3}{2}}) + \vartheta(h_J^{r_0-\varepsilon}))\|v\|_{1,2}. \tag{8}$$

where h and h_J are diameters of the largest and the smallest element in K^h, respectively.

Proof. See [8].

From Lemma 2 and Theorem 3.1.5 in [9] we can prove Lemma 3.

Lemma 3. Let (λ_h, u_h) be the *k*-th bi-linear eigenpair of (1), and $\|u_h\|_{1,2} = 1$, then there exists the *k*-th eigenpair (λ, u) of (1) such that

$$\|u_h - I_h u\|_{1,2} \le C(h^{\frac{3}{2}} + h_J^{r_0-\varepsilon}). \tag{9}$$

Proof. See [9].

Remark 1. The above Lemma 2 and Lemma 3 are not only suitable for the singularity caused by domains with reentrant corners, but also suitable for the singularity caused by coefficients too. For membrane vibrating problem on *L*-shaped region, Lemma 2 and Lemma 3 are valid because in this situation the singularity is caused by domains with reentrant corners, and $r_0 = \frac{2}{3}$.

Denote $\|v\|_h = \left(\sum_{k \in K^{2h}} |v|_{1,2,k}^2\right)^{\frac{1}{2}}$.

Theorem 1. The bi-quadratic interpolation $I_{2h}^2 u_h$, obtained by Scheme 1, for bi-linear element has the sup-convergence, i.e. there exists $u \in V_\lambda$, such that

$$\|I_{2h}^h u_h - u\|_h \le C(h^{\frac{3}{2}} + h_J^{r_0-\varepsilon}), \tag{10}$$

Proof. From (8) and (9), we get

$$\|u_h - P_h u_h\|_h \le C(h^{\frac{3}{2}} + h^{r_0-\varepsilon}).$$

By interpolation processing, we obtain

$$\|I_{2h}^2 u_h - I_{2h}^2 P_h u_h\|_h \le C\|u_h - P_h u_h\|_h \le C(h^{\frac{3}{2}} + h^{r_0-\varepsilon}).$$

From (8) we can deduce $\|I_{2h}^2 P_h u_h - u\|_h \le C(h^{\frac{3}{2}} + h^{r_0-\varepsilon})$, so

$$\|I_{2h}^2 u_h - u\|_h \le \|I_{2h}^2 u_h - I_{2h}^2 P_h u_h\|_h + \|I_{2h}^2 P_h u_h - u\|_h \le C(h^{\frac{3}{2}} + h^{r_0-\varepsilon}).$$

Therefore, we get (10).

If $I_{2h}^2 u_h \in H_0^1(\Omega)$, then, let $w = I_{2h}^2 u_h, \lambda_r = a(w, w) \big/ \|w\|_{0,2}^2$ in the fundamental relationship(6)and we can derive

$$\lambda_r - \lambda \le \|I_{2h}^2 u_h - u\|_{1,2}^2 \Big/ \|I_{2h}^2 u_h\|_{0,2}^2 \le C(h^{\frac{3}{2}} + h^{r_0-\varepsilon})^2. \tag{11}$$

From Theorem 1 we know that when the number of encryption is enough, i.e. $h_J^{r_0-\varepsilon} = \vartheta(h^{\frac{3}{2}})$, the accuracy order of the bi-quadratic interpolation for the eigenfunciton of bi-linear element is $\vartheta(h^{\frac{3}{2}})$ in energy norm.

4. The scheme of bi-quadratic interpolation with hanging nodes in locally refined region

In this section, we establish two schemes. The difference between those two schemes is in step 3. In step 3 of Scheme 1, the value of piecewise polynomial $I_{2h}^2 u_h$ at a hanging node is the average of the values of $u_h \in S_h$ at its two adjacent regular nodes. Thus, we can not guarantee that $I_{2h}^2 u_h$ is continuous at hanging nodes, i.e. $I_{2h}^2 u_h \notin H_0^1(\Omega)$. So the resulting Rayleigh quotient λ_r computed by this $I_{2h}^2 u_h$ does not meet (11), moreover, we cannot write $a(I_{2h}^2 u_h, I_{2h}^2 u_h)$ but $a_h(I_{2h}^2 u_h, I_{2h}^2 u_h)$, the sum of integral on element.

For reason above, we establish Scheme 2. Step 3 of Scheme 2 is based on the discussion made on the bi-quadratic interpolation with hanging nodes in the third section: $I_{2h}^2 u_h$ is continuous at hanging nodes A_2, A_4 (see Fig. 2. (b)) and must satisfy the restrained condition (6). So, we modify the value of $I_{2h}^2 u_h$ at hanging nodes A_2, A_4 to ensure that $I_{2h}^2 u_h$ is continuous at A_2, A_4 .

But unfortunately, we are not able to give the proof on the order of convergence for $I_{2h}^2 u_h$ and λ_r obtained by Scheme 2, which is our future work.

Scheme 1.

Step 1: Generate a locally refined regular mesh K^{2h} with hanging nodes in refined mesh.

Step 2: Subdivide each element in K^{2h} into four identical rectangles to get the mesh K^h . Solve (3) on K^h with bilinear rectangle element, and the eigenvalue λ_h and eigenfunction u_h are obtained.

Step 3: Make bi-quadratic interpolation on K^{2h} with the values of u_h at regular nodes and the values of u_{2h} at hanging nodes in K^h which are modified as follows.

See Fig 2. (b), A_2 and A_4 are hanging nodes, the values of u_{2h} at A_2 and A_4 are respectively for

$$u_2 = \frac{u_1 + u_3}{2}, u_4 = \frac{u_3 + u_5}{2}.$$

Step 4: Calculate the Rayleigh quotient:

$$\lambda_h = a_h(I_{2h}^2 u_h, I_{2h}^2 u_h) \Big/ \left\| I_{2h}^2 u_h \right\|_{0,2}^2 .$$

Scheme 2.

Step 1 and *Step* 2 are the same as Scheme 1.

Step 3: Make bi-quadratic interpolation on K^{2h} with the values of u_h at regular nodes and the values of u_{2h} at hanging nodes in K^h which are modified as follows.

See Fig 2. (b), A_2 and A_4 are hanging nodes, the values of u_{2h} at A_2 and A_4 are respectively for

$$u_2 = \frac{3}{8}u_1 + \frac{3}{4}u_3 - \frac{1}{8}u_5, u_4 = -\frac{1}{8}u_1 + \frac{3}{4}u_3 + \frac{3}{8}u_5.$$

Step 4: Calculate the Rayleigh quotient:

$$\lambda_h = a(I_{2h}^2 u_h, I_{2h}^2 u_h) \Big/ \left\| I_{2h}^2 u_h \right\|_{0,2}^2 .$$

5. Numerical experiment

Consider the membrane vibrating problem (1) on the *L*-shaped region $\Omega = [0,1]\times[0,1]-[\frac{1}{2},1]\times[\frac{1}{2},1]$. We use the bi-quadratic interpolation post-processing of bilinear rectangular element, adopting the refinement procedure described in the second section, to compute the smallest eigenvalue of (1) by Scheme 1 and Scheme 2, respectively. The experiment results are listed in Table 1, which indicate that Scheme 2 has a high efficiency.

Table1: The results of calculating by Scheme 1 and Scheme 2

H	J	λ_H	λ_{r1}	$\lvert\lambda_{r1}-\lambda\rvert$	λ_{r2}	$\lvert\lambda_{r2}-\lambda\rvert$
$\frac{\sqrt{2}}{8}$	4	39.441050	39.0420770	0.483197	38.689551	0.130671
$\frac{\sqrt{2}}{16}$	5	38.775173	38.6580205	0.099140	38.569139	0.010259
$\frac{\sqrt{2}}{32}$	6	38.612356	38.5819619	0.023082	38.559818	0.000938
$\frac{\sqrt{2}}{64}$	7	38.572174	38.5645044	0.005624	38.558993	0.000113
$\frac{\sqrt{2}}{128}$	9	38.562197	38.5602783	0.001398	38.558904	0.000024

In Table 1 we use H and J to denote the diameter of the initial mesh and the number of encryptions, respectively. Let λ_H denote the eigenvalue on the initial mesh, λ_{r1} and λ_{r2} are the Rayleigh quotients calculated by Scheme 1 and Scheme 2, respectively.

Acknowledgments

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